A state-space description of transfer effects in isomorphic problem situations

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Previous research has shown significant transfer for subjects solving two problems of isomorphic structure. The state-space representation of these problems is used to characterize their structure, and in particular, the state-space decomposition *modulo* isomorphic sub-problems is used to define "stages" within the problem's solution. Changes within these "stages" provide a framework for a more complete description and understanding of the transfer phenomenon.

1. Introduction

Recent research (Luger, 1976; Goldin & Luger, 1978; Luger & Steen, 1978) has demonstrated how the state-space representation of the Tower of Hanoi problem (TOH) can be used to describe the effects of the sub-problem and symmetry structure of the problem on problem solver's behaviour. The TOH problem has been extensively described in the literature (Simon, 1975; Luger, 1976) and its state-space considered by Nilsson (1972). It is a natural problem to consider for human problem solving analysis because its well defined state-space has a rich sub-problem structure and possesses somewhat more symmetry than is immediately apparent in the problem.

In the TOH problem four concentric rings, labelled 1, 2, 3 and 4 to indicate their respective size, are placed in order of size with the largest on the bottom on the first of three pegs, labelled A, B and C. The apparatus is pictured in Fig. 1(a). The object of the problem is to transfer all the rings from peg A to peg C in the minimum number of moves. Only one ring may be moved at a time and no larger ring placed over a smaller ring on any peg.

Figure 1(b) gives the complete state-space representation of the TOH problem, each circle standing for a possible position or state of the game. The four letters labelling a state refer to the respective pegs on which the four rings are located. For example, state BBBA means that rings 1, 2 and 3 are in their proper order on peg B while ring 4 is on peg A. A legal move by the problem solving subject always effects a transition between states represented by neighbouring circles in the state-space. The solution path containing the minimum number of moves consists of the 15 steps from AAAA to CCCC down the right side of the state-space diagram.

The TOH has a natural decomposition into nested sub-problems. For example, to solve the 4-ring TOH problem it is necessary at some point to move the largest ring from its position on peg A to peg C, but before this can be done the three smaller rings must be assembled in their proper order on peg B. This problem of moving the three rings from one peg to another may be termed a 3-ring sub-problem of the 4-ring TOH. The

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3-ring sub-space constitutes a subset of the state space of the 4-ring TOH problem. The 4-ring TOH state space contains three 3-ring sub-spaces, differing by reason of the position of ring 4. Each sub-space becomes a sub-problem when one of its entry states is designated as an initial state, and its exit states are designated as goal states. Similarly each 3-ring sub-space contains three 2-ring spaces for a total of nine in the 4-ring TOH state-space. Each 2-ring sub-space may be further decomposed into three 1-ring sub-spaces. Note examples of sub-spaces in Fig. 1(b).
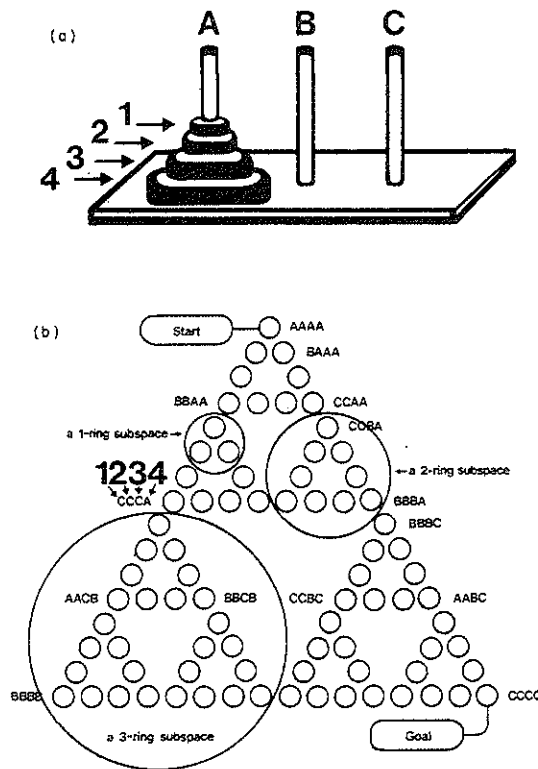

FIG. 1. (a) The Tower of Hanoi problem in the "start" state. (b) The state-space representation of the Tower of Hanoi and Tea Ceremony problems. Note examples of 1-, 2- and 3-ring sub-spaces.

It should be realized that the structure of moves within each n-ring sub-problem is identical, that is, even though the "start" and "goal" pegs, and the position of the rings larger than n may differ in each instance, a one-to-one onto mapping exists which preserves the set of possible legal moves in each sub-problem, i.e. each move within one subproblem corresponds to one and only one move in the second subproblem, and conversely. In this sense, all TOH sub-problems of n rings (for fixed n) are said to be isomorphic. Furthermore, each n-ring sub-problem, as well as the entire problem admits of a symmetry mapping. This automorphism (one-to-one mapping of the space onto itself) maps a goal state of the n-ring problem onto its conjugate goal state, which

corresponds to transferring the n rings to the other open peg. Were the three pegs of the TOH board at the corners of an equilateral triangle (as are the people in the Tea Ceremony, to be discussed) the symmetry automorphism would represent the geometric operation of reflection about the altitudes of the triangle.

Problem solvers' behaviour may then be described as paths through the state-space representation of the TOH problem. In Luger (1976), these paths were seen as non-random and goal directed through the problem. Subgoal states were used by the subjects in a unique fashion. First, the subjects' paths within the sub-problem space were sub-goal directed. Second, whenever a sub-goal state was entered, subjects exited from the state in such a manner as to also exit from the sub-problem. This occurred 96% of the time for 2-ring sub-problems and 98% of the time for 3-ring sub-problems, strong evidence (since random exit, excluding backups, would be 50%) that these states were, in effect, used as sub-goal states by the subjects.

Furthermore, subjects displayed the effects of the problem's symmetry. This they did by producing consecutive path segments congruent *modulo* a symmetry automorphism of the state-space. The existence of congruent pairs of paths was correlated with subjects first moves in solving the problem, occurred most often after a 2- or 3-ring sub-problem "stage" (see below), and more often than other paths, led directly to the solution of the problem (Goldin & Luger, 1978).

Finally, "stages" or "episodes" in the problem's solution were identified wherein minimal path solutions were generated for isomorphic sub-problems within the problem. "Stages" occurred when subjects executed minimal solutions for n-ring sub-problems for a definable period before executing minimal solution to $(n+1)$-ring sub-problems. The "stage" aspect of subjects' solution is the major focus of this paper and will be discussed further in the following section.

In the Luger & Bauer research the effects on problem solver's behaviour of solving two different problems of isomorphic structure were considered. In that study two different problems of identical structure were given to subjects in the traditional test for transfer. Both the number of states entered by subjects' problem solving paths as well as the elapsed times required for solution were measured. An analysis of variance on time required and states entered demonstrated significant transfer effects for each problem order.

To be more specific, a second problem called the Tea Ceremony (Hayes & Simon, 1976) was given to each subject. The Tea Ceremony (TC) pictured in Fig. 2, consists of three people (a host, an elder, and a youth) performing four tasks (feeding the fire, serving cakes, serving tea and reading poetry). These services have a priority order, represented by the seriated blocks and are transferred, one at a time, from the host (at the start of the ceremony) to the youth (at which time the ceremony is completed). The transferring is accomplished by sliding the blocks along the "track" in Fig. 2. It is not permitted for any person to take a more important task while already possessing a lesser task (this would be represented by taking a block larger than any block already possessed).

It may be checked that the TOH and TC problems are isomorphic: the three people corresponding to the three pegs, and the four services corresponding to the four rings. There is a one-to-one onto mapping of legal moves and solution paths. Thus both TOH and TC are represented by the same state-space (Fig. 1(b)) and possess the same sub-problem and symmetry decompositions.
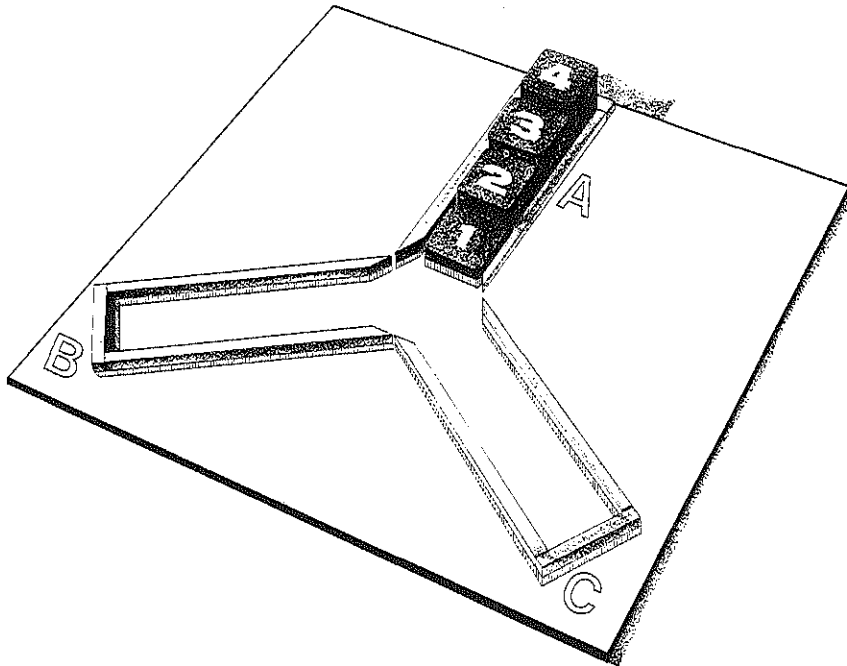
FIG. 2. The Tea Ceremony Problem in its initial state. 1, 2, 3, 4, A, B, C illustrate the TOH-TC isomorphism. For testing purposes, FIRE, CAKES, TEA, POETRY, HOST, ELDER and YOUTH replace 1, 2, 3, 4, A, B and C, respectively.

In the transfer experiment two groups of subjects solved the TOH and TC problems, one group the TOH first (TOH1-TC2) and the second group the TC first (TC1-TOH2). The subjects' behaviour on TC1 was then compared with their behaviour on TC2, and behaviour on TOH1 with that on TOH2.

The focus of this paper is to demonstrate how the decomposition of the TOH/TC state-space may be used to characterize the transfer effects of the TOH/TC transfer study. This characterization will go well beyond the traditional statistical demonstration of effects within a population of subjects to indicate a possible explanation of these effects.

2. "Stages" or "episodes" in TOH/TC solutions

An n-ring stage occurs in solving the TOH/TC problems when a subject executes minimal solutions to 50% or more of all the n-ring sub-problem isomorphs for a certain time period (at least the length of an $(n+1)$-ring sub-problem) prior to executing minimal solutions to 50% or more of the $(n+1)$-ring sub-problem isomorphs.

To understand the motivation of this stage characterization, consider a 2-ring sub-problem of the TOH problem. As indicated above the 4-ring TOH state-space may be decomposed into nine 2-ring sub-spaces. Each of these nine sub-spaces has three different possible "start" states, bringing to twenty-seven the number of possible environmental situations (with twenty-seven sets of perceptual cues) for the 2-ring

sub-problem of the 4-ring TOH board. Furthermore, considering the different possible paths through the twenty-seven different 2-ring sub-problems, the environmental situations a problem solver could possibly confront is seen to be of exponential proportions. Yet the structures of these possible move situations are identical, and may be represented as the 2-ring sub-problem space.

The problem solver asked to find a minimal solution for the 4-ring TOH problem must find solutions to these 2-ring sub-problems. When the solver accomplishes this—that is, produces a sequence of minimal paths for each 2-ring sub-problem encountered, two equivalent characterizations of his/her activity are possible: (1) the solver may be said to decompose the 4-ring problem space *modulo,* its 2-ring sub-problems; and (2) with respect to 2-ring sub-problems, the solver may be said to have detected an invariant of the problem or performed a "conservation". That is, the solver has responded identically to (behaviourly, by producing minimal paths through) each of these perceptually different environmental situations.
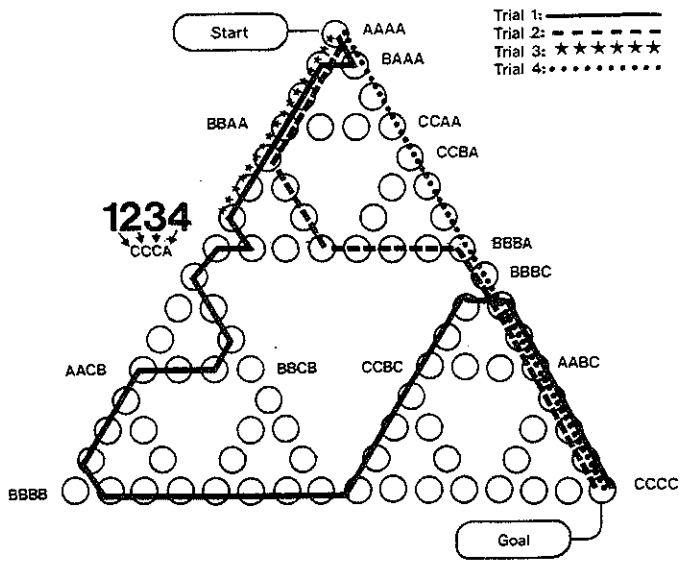
	Stage 1		Stage 2		Stage 3		Solution	
T	10	105		230	320	360		
S	3	22		52	67	83		
2-ring	0 0 0 0	1 1 0 1 1 1 1		1 1 1 1		1 1 1 1			
3-ring	0 0 0		0	1	1	1 1		
4-ring	0		0		0	1		

FIG. 3. The behaviour of one subject solving the Tea Ceremony problem. *T* marks the amount of time in seconds elapsed and *S* the number of states entered to the beginning of each stage of the solution. *0* indicates a non-minimal and *1* a minimal path through the 2-, 3- and 4-ring subproblems.

Two empirical facts need to be emphasized in studying the decomposition/conservation of several problem solving populations (undergraduate psychology majors and college educated adults) (1) the stage phenomenon was fairly general. In (Luger, 1976) 88% of the subjects went through at least one stage, 57% of the subjects showed at least two stages, and 13% showed all three theoretically possible stages corresponding to the 1-, 2- and 3-ring sub-problems of the 4-ring TOH problem. (2) The sub-problem decomposition/conservation occurs in an "all or nothing" fashion. That is, although a 50% minimal path solution was given as the *a priori* criterion for determining the existence of stages, the actual per cent of minimal solutions was much lower before the start of the stage and much higher after the start of the stage. In the 2-ring stages, for example, 84% of the subjects had less than 33% minimal paths before, and 95% of the subjects had more than 80% minimal paths (with 87% of the subjects actually having 100% minimal paths) after the 2-ring stage had begun.

An analysis similar to that above for 2-ring sub-problems, may be carried out for the other isomorphic sub-problem situations within the TOH/TC problem. The occurrence of the sequence of stages within the problem's solution is an attempt to monitor the occurrence of the sub-problem decompositions/conservations. To further clarify these notions, the behaviour of a subject solving the TC problem is examined in Fig. 3.

The subject of Fig. 3 was selected because she exhibited all three theoretically possible stages in coming to the solution of the 4-service TC problem. The "T" and "S" in Fig. 3 mark the time elapsed in seconds, and the number of states entered to the beginning of each stage. A "1" indicates a minimal solution path of a sub-problem, while an "0" indicates the subject took a non-minimal path through the sub-problem space. The 2-, 3- and 4-ring sub-problems are placed above each other in a manner to indicate (left to right) the time sequence of problem spaces entered. The top to bottom alignment indicates the simultaneity of problem spaces. Note that in trial 1, three 3-ring sub-problems were entered, the first 3-ring sub-problem contained two 2-ring sub-problems, the second 3-ring sub-problem contained three 2-ring sub-problems, and so on.

3. The description of transfer effects and hypotheses tested

In this section two aspects of solution stages within transfer effects will be presented. First, a description of the stage effect in each group of subjects. This description includes both the number of subjects spending time within each stage for each group as well as the average amount of time each group spends in each stage. Second, two sets of statistical tests are presented to test the significance of the stage effects. The first tests compare the stage effect of transfer, that is, compares TC1 with TC2 and TOH1 with TOH2- while the second tests analyse the effects of solving two consecutive different problems of isomorphic structure.

The goal of this study is to demonstrate that the significant improvement in problem solving performance that was evident in the Luger & Bauer study may be further clarified by analyzing the stages in solution. It is hypothetesized that the time spent within stages will (1) not increase for any stage in the second problem and (2) will decrease significantly for one or more of the stages in the second problem. Furthermore,

it is hypothesized that (1) and (2) above will again be true in the analysis of transfer across two different problems of isomorphic structure.

Table 1 presents the number of subjects in each group that show the stage effect. The 1-, 2- and 3-ring sub-problem stages are considered for each problem and group. The average amount of time (in seconds) spent by each group of subjects in each stage is presented.

TABLE 1

A description of the number of subjects, N and the average time, A (in seconds), of all 19 subjects within each stage of each problem

Stage		TOH1	TOH2	TC1	TC2
1	N	8	4	7	3
	A	43·1	12·9	75·1	12·3
2	N	19	11	16	17
	A	199·1	68·4	247·1	187
3	N	8	4	7	6
	A	32·6	20·3	46·7	48·4

In Table 2 the results of the one-tailed Mann-Whitney "U" test are presented for each problem in the test for transfer. The "U" test determines: (a) whether the decrease in time (therefore, one-tailed) to a particular point in the solution (the beginning of each stage) is significant; and (b) whether the decrease in time (one-tailed, again) *within* each stage is significant.

Table 3 presents the transfer effects of solving two consecutive different problems of isomorphic structure. The sign test is used because the samples are related. That is, the times to each stage and within each stage are compared for the first and second problems of each subject. The sign test calculates the total number of increases or

TABLE 2

The analysis of transfer effects up to and within each stage

	Stage	TOH1—TOH2	Sig.	TC1—TC2	Sig.
A	1	U = 161·5	NS	U = 152	NS
	2	U = 131	NS	U = 130·5	NS
	3	U = 61	$P < 0.001$	U = 110	$P < 0.025$
	Solution	U = 63	$P < 0.001$	U = 116·5	$P < 0.05$
B	1	U = 140·5	NS	U = 132·5	NS
	2	U = 76·5	$P < 0.001$	U = 125·5	$P < 0.053$
	3	U = 143·5	NS	U = 172·5	NS
	Solution	U = 175	NS	U = 129	NS

The one-tailed Mann-Whitney "U" test is used to determine the significance of time decreases: A, up to: and B, within each stage for each problem.

decreases of time at or during each stage and compares these increases and decreases with those of two random samples. The one-tailed test is used because it is hypothesized that the time up to and/or within stages will decrease. In Tables 2 and 3 the *time to* stage *n* indicates the elapsed time to the start of that stage or production of minimal step solution paths for the *n*-ring sub-problem; and *time within* indicates the elapsed time during which the subject produces the minimal path solution for the *n*-ring sub-problem before producing minimal paths for sub-problems of more than *n* rings.

TABLE 3

The analysis of time changes in solving two problems of isomorphic structure

Stage	TOH1 and TC1	TC2 and TOH2	Sig.
A 1	$T = 7$	$D = 6$	NS
2	$T = 21$	$D = 17$	$P < 0.005$
3	$T = 38$	$D = 28$	$P < 0.01$
Solution	$T = 38$	$D = 29$	$P < 0.01$
B 1	$T = 21$	$D = 16$	$P < 0.01$
2	$T = 38$	$D = 27$	$P < 0.01$
3	$T = 20$	$D = 13$	NS
Solution	$T = 34$	$D = 21$	NS

The one-tailed sign test for related samples is used to determine the significance of the number of time decreases. The analysis is A, up to; and B, within each stage. T is the total subjects (of 38) showing time changes and D the number of decreases.

4. Results and conclusions

Luger & Bauer (1978) have shown in previous research significant transfer effects in groups of subjects solving two problems of isomorphic structure. The goal of this paper is to analyze structural features of these problem isomorphs in an attempt to further explain these transfer effects. The notion of "stages" within the solution of these isomorphic problems was discussed in section 2 and made up the framework for the hypotheses of section 3.

Table 1 described the occurrences and durations of the stage effect. The percentages of subjects spending time within intervals confirms the findings of earlier studies with more than 88% of the subjects experiencing at least one stage, 47% at least two stages and 9·2% all three theoretically possible stages. Note that only about 40% of the subjects experienced stage 1 in the first problem (TOH1 or TC1), while less than half that mount, 18·4%, experienced stage 1 in the second problem (TOH2 or TC2). Note also the decrease in average time spent in stage 1 between the first and second problems for all subjects (TOH 43·1 to 12·9 seconds and TC 75·1 to 12·3 seconds).

More than 92% of all subjects spent some time in stage 2 while solving their first problem. Although this number was not radically decreased (73·7%) in going from the first to the second problem, the changes in time spent within stage 2 do mark the major evidence of transfer effects in the problem solving, (TOH 199·1 to 68·9 seconds and TC 247·1 to 187 seconds). Finally, only 40% of all subjects solving the first problem spent

time in stage 3 with this decreasing to 28·9% for problem two. The average time spent for all subjects in this stage remained virtually unchanged.

The frequency and distribution of stages in problem solution may also be ascertained from Table 1. Three things may be noted: (1) the distribution of stages remains identical across both problem type and problem order; (2) the frequency of stages decreases by about 50% for the TOH problem type; and (3) the frequency, except for stage 1, remains constant for the TC problem type.

Table 2 describes the significance of transfer effects up to and within each stage. The only significant within stage transfer occurred, for both TOH and TC, in stage 2. It may be observed that up to stage 2 there had been no significant transfer, and after stage 2 (up to and including the production of the minimal step solution path) there are no major changes in the significance levels.

Table 3 describes the results of the sign test of related samples across two different problems of isomorphic structure. Here behaviour is compared for each subject between the first (TOH1, TC1) and the second (TOH2, TC2) problems encountered. This test compares only the increases and decreases of time spent to or within each stage. This eliminates the possibilities of large variances in individual subjects behaviour from clouding the results across all subjects. Of 38 subjects, 21 experienced stage 1; for 16 of these the time in stage 1 decreases between the first and second problem. This is significant at the 0·02 level. All 38 subjects spend time in stage 2, and for 27 of these ($P < 0·01$) the time decreased in the second problem. Stage 3 and the time of solution show no significant decrease between the first and second problem. In fact, in no situation was there significant difference between the first and second problem in time required to perform the 15 step minimal solution path. These facts seem to indicate that the transfer phenomenon is not a "speeded up" performance on a fixed set of moves but rather an improved ability to decompose the problem by performing minimal solutions across its isomorphic sub-problems.

The cumulative effect of the changes in time can be seen in Table 3, section A, where there is no improvement up to stage 1; there are large increases in the number of subjects showing time changes up to stage 2 (from 7 to 21 subjects) and up to stage 3 (from 21 to 38 subjects). It should be noted that the sign test measures the significance of the number of time decreases (one-tailed) as a proportion of subjects showing time changes. Therefore the percentage of subjects showing any difference in times must also be considered. Thus, 21 of 38 subjects showed time changes in stage 1, all 38 showed changes in stage 2, and 20 of the 38 subjects showed time changes in stage 3.

The data seem to verify the hypotheses of the previous section. In the transfer comparison (TOH1—TOH2 and TC1—TC2), there was not a significant time decrease in stage 1 for all subjects. But, as noted above, while 40% of the subjects experienced stage 1 in the first problem presented, only 18% experienced it in the second problem, and the average time in stage 1 of all subjects was down by more than 75% for the second problem. For stage 2 there was a significant time decrease for both problems. Stage 3 and the solution showed no significant time decreases. There were no time increases for any stage or problem. In the comparison of the first with the second problem faced, using the sign test for related samples, there was a significant decrease in time in both stages 1 and 2, and no significant increase within any stage.

Stage 3 did not show significant transfer effects in any of the problem situations. This is possibly because this stage was immediately prior to solution and not "embedded" in

the problem as deeply as was stage 2. In fact, with the first minimal state solution path of the 4-ring problem within a 90-second period the task was concluded, and so the 3-ring stage usually ended with only one success of the 4-ring problem. It may be hypothesized that a transfer study on the 5-ring TOH/TC would show much larger transfer in a stage 3 situation, and perhaps, again, much less transfer in "stage 4". Furthermore, the stage effect should be studied in further isomorphically related tasks, including transfer between two different tasks on the same problem.†

In understanding the data of this study the theoretical underpinning of the "stage" conjectures must be remembered. This is that the solution of the TOH/TC problems may be seen as the decomposition of the state space of the problem *modulo* its isomorphic 1-, 2- and 3-ring sub-problems. This decomposition corresponds to a "conservation" or recognition of invariance by the problem solving subject in iso-morphic n-ring sub-problems for fixed n; these isomorphic sub-problems are respon-ded to in a unique fashion, with the production of a minimal step solution path.

Finally, the data of this study seems to indicate that human performance on the TOH and TC problems is very unlike the GPS model for solving problems in this domain (Ernst & Newell, 1969; Luger, 1976). Rather than an analysis and ordering of "differences" in problem goals with the subsequent solution of the largest differences first, the human subject seems to work out solutions for smaller differences first and then propogate these solutions by recognition of invariances or "conservations" across the isomorphic sub-structures of the problem. In this sense the subject may be said to "decompose" the problem.

In the analysis of transfer effects, the human—GPS dichotomy is even wider. GPS must, in solving the second problem, recognize and order differences and again attempt to reduce the largest difference first. That is, it proceeds exactly as it did with the first problem faced. The human solving the second problem, decomposes it by discovering its invariances. This may happen more quickly for the second problem as it did in this study, since the invariances for both problems are related.

The state-space representation has been used throughout this study to characterize these invariances. The paths through the state space have been analysed to monitor the subjects' decomposition or recognition of invariance within each problem situation.

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† In preliminary analysis of a subsequent study (Luger & Steen, 1978), where subjects were given two different tasks with the Tower of Hanoi problem, it was found that subjects' stages in the first task were much the same as in the first tasks of this study, but that subjects began the second task at stage 3 ($P < 0.05$ sign tests). Thus, in this situation, the 1- and 2-ring subproblem invariants were "transferred" to the second problem solving task.

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